Existence of Stationary Distributions for Markov Chains with Infinite State Spaces

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1 Introduction

The existence of stationary distributions for Markov chains with finite state spaces is relatively straightforward to establish. However, when dealing with infinite state spaces, additional challenges arise. This document explores the conditions under which stationary distributions exist for Markov chains with countably infinite state spaces, with a particular focus on the Foster-Lyapunov criterion.

2 Preliminaries

We begin by establishing key definitions that will be essential for our analysis.

Definition 1 (Markov Chain). *A sequence of random variables $\{X_n\}_{n\geq 0}$ with state space $\mathcal{S}$ is a Markov chain if for all $n \geq 0$ and all states $i_0, i_1, \ldots, i_n, j \in \mathcal{S}$:*

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_{n+1} = j \mid X_n = i_n). \quad (1)$$

For a time-homogeneous Markov chain, the transition probabilities are given by:

$$P(i,j) = P(X_{n+1} = j \mid X_n = i), \quad (2)$$

which are independent of n. We denote the matrix of transition probabilities as $P = (P(i,j))_{i,j\in\mathcal{S}}$.

Definition 2 (Irreducibility). *A Markov chain is irreducible if for all $i, j \in \mathcal{S}$, there exists an integer $n > 0$ such that $P^n(i,j) > 0$, where $P^n(i,j)$ is the n-step transition probability from state i to state j.*

Definition 3 (Recurrence and Transience). *For a state $i \in \mathcal{S}$, define the return time as $T_i = \inf\{n \geq 1 : X_n = i \mid X_0 = i\}$. A state i is:*

- *Recurrent if $P(T_i < \infty) = 1$.*
- *Positive recurrent if $E[T_i] < \infty$.*
- *Null recurrent if $P(T_i < \infty) = 1$ but $E[T_i] = \infty$.*

- *Transient if* $P(T_i < \infty) < 1$.

In an irreducible Markov chain, either all states are recurrent or all states are transient. Furthermore, if the chain is recurrent, either all states are positive recurrent or all states are null recurrent.

3 Stationary Distributions

Definition 4 (Stationary Distribution)**.** *A probability distribution* $\pi = (\pi_i)_{i \in \mathcal{S}}$ *on the state space* $\mathcal{S}$ *is called a stationary distribution if:*

$$\pi P = \pi, \tag{3}$$

or equivalently, for all $j \in \mathcal{S}$*:*

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i P(i, j). \tag{4}$$

The following theorem establishes the connection between positive recurrence and stationary distributions:

Theorem 1. *For an irreducible Markov chain:*

1. *If the chain is positive recurrent, then there exists a unique stationary distribution* π.
2. *If the chain is null recurrent or transient, then no stationary distribution exists.*

Furthermore, if the chain is positive recurrent, the stationary distribution is given by:

$$\pi_i = \frac{1}{E[T_i]}, \quad \text{for all } i \in \mathcal{S}. \tag{5}$$

The challenge with infinite state spaces is determining whether a chain is positive recurrent. The Foster-Lyapunov criterion provides a powerful tool for this purpose.

4 Foster-Lyapunov Criterion

The Foster-Lyapunov criterion provides sufficient conditions for positive recurrence.

Theorem 2 (Foster-Lyapunov Criterion)**.** *Consider an irreducible Markov chain* $\{X_n\}_{n \geq 0}$ *with countable state space* $\mathcal{S}$ *and transition matrix* P*. If there exists a function* $V : \mathcal{S} \to [0, \infty)$*, a finite set* $C \subset \mathcal{S}$*, and constants* $\epsilon > 0$ *and* $b < \infty$ *such that:*

$$E[V(X_{n+1}) - V(X_n) \mid X_n = i] \leq \begin{cases} -\epsilon, & \text{if } i \notin C \\ b, & \text{if } i \in C \end{cases} \tag{6}$$

then the Markov chain is positive recurrent.

The function V is often called a Lyapunov function. Intuitively, the criterion states that, on average, the value of V decreases outside a finite set C, ensuring that the chain returns to C regularly.

Remark 1. *An alternative formulation of the Foster-Lyapunov criterion is: There exists a function $V : \mathcal{S} \to [0, \infty)$, constants $\lambda \in (0, 1)$, and $b < \infty$ such that:*

$$E[V(X_{n+1}) \mid X_n = i] \leq (1 - \lambda)V(i) + b\mathbf{1}_C(i), \quad \textit{for all } i \in \mathcal{S}, \tag{7}$$

where $\mathbf{1}_C(i)$ is the indicator function for the set C.

5 Interpreting the Drift Conditions

The Foster-Lyapunov criterion can be interpreted in terms of drift conditions. The term:

$$\Delta V(i) = E[V(X_{n+1}) - V(X_n) \mid X_n = i] \tag{8}$$

represents the expected change (or drift) in the Lyapunov function V when the chain is in state i.

The Foster-Lyapunov criterion requires that this drift be consistently negative outside a finite set C, creating a "potential well" that prevents the process from escaping to infinity.

6 Applications to Specific Markov Chains

6.1 Birth-Death Processes

A birth-death process is a Markov chain on $\mathcal{S} = \{0, 1, 2, \ldots\}$ with transitions limited to neighboring states:

$$P(i, i + 1) = p_i \quad \text{(birth probability)} \tag{9}$$

$$P(i, i - 1) = q_i \quad \text{(death probability)} \tag{10}$$

$$P(i, i) = r_i = 1 - p_i - q_i \quad \text{(probability of no change)} \tag{11}$$

where $q_0 = 0$ (no deaths from state 0).

Theorem 3. *An irreducible birth-death process is positive recurrent if and only if:*

$$\sum_{i=1}^{\infty} \frac{p_0 p_1 \cdots p_{i-1}}{q_1 q_2 \cdots q_i} < \infty. \tag{12}$$

Furthermore, if positive recurrent, the stationary distribution is given by:

$$\pi_0 = \left(1 + \sum_{i=1}^{\infty} \frac{p_0 p_1 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}\right)^{-1}, \tag{13}$$

$$\pi_i = \pi_0 \frac{p_0 p_1 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}, \quad \textit{for } i \geq 1. \tag{14}$$

6.2 M/M/1 Queue

The M/M/1 queue is a continuous-time Markov chain that models a single-server queue with Poisson arrivals (rate λ) and exponential service times (rate μ). When embedded at transition times, it becomes a discrete-time birth-death process with:

$$p_i = \frac{\lambda}{\lambda+\mu}, \quad \text{for all } i \geq 0, \tag{15}$$

$$q_i = \frac{\mu}{\lambda+\mu}, \quad \text{for all } i \geq 1, \tag{16}$$

$$r_i = 0, \quad \text{for all } i \geq 0. \tag{17}$$

Example 1 (M/M/1 Queue)**.** *We can verify that the M/M/1 queue is positive recurrent when $\lambda < \mu$ using the Foster-Lyapunov criterion.*

Choose $V(i) = i$ (the queue length) and compute the expected drift:

$$\begin{aligned}
\Delta V(i) &= E[V(X_{n+1}) - V(X_n) \mid X_n = i] && (18)\\
&= p_i(i+1) + q_i(i - 1\mathbf{1}_{i>0}) - i && (19)\\
&= \frac{\lambda}{\lambda+\mu}(i+1) + \frac{\mu}{\lambda+\mu}(i - 1\mathbf{1}_{i>0}) - i && (20)\\
&= \frac{\lambda - \mu + \mu\mathbf{1}_{i=0}}{\lambda+\mu} && (21)
\end{aligned}$$

When $i > 0$ and $\lambda < \mu$, we have $\Delta V(i) = \frac{\lambda-\mu}{\lambda+\mu} < 0$. This satisfies the Foster-Lyapunov criterion with $C = \{0\}$, establishing positive recurrence.

Using the formula for birth-death processes, the stationary distribution is:

$$\pi_i = (1-\rho)\rho^i, \quad \textit{for } i \geq 0, \tag{22}$$

where $\rho = \frac{\lambda}{\mu} < 1$ is the traffic intensity.

6.3 Random Walk on $\mathbb{Z}$

Consider a simple random walk on $\mathbb{Z}$ with:

$$P(i, i+1) = p, \tag{23}$$

$$P(i, i-1) = q = 1 - p. \tag{24}$$

Proposition 4. *The simple random walk on $\mathbb{Z}$ is:*

- *Transient if $p \neq q$.*
- *Null recurrent if $p = q = 1/2$.*

Therefore, it does not possess a stationary distribution in either case.

This example illustrates that even seemingly simple Markov chains with infinite state spaces may not have stationary distributions.

7 Additional Conditions for Positive Recurrence

Beyond the Foster-Lyapunov criterion, several other sufficient conditions can establish positive recurrence:

Theorem 5 (Geometric Foster-Lyapunov)**.** *If there exists a function $V : \mathcal{S} \to [1, \infty)$, constants $\lambda \in (0, 1)$, and $b < \infty$ such that:*

$$E[V(X_{n+1}) \mid X_n = i] \leq \lambda V(i) + b\mathbf{1}_C(i), \quad \textit{for all } i \in \mathcal{S}, \tag{25}$$

where C is a finite set, then the Markov chain is geometrically ergodic, i.e., it converges to its stationary distribution at a geometric rate.

Theorem 6 (Atoms and Small Sets)**.** *If an irreducible Markov chain has an atom (a state or set of states with special recurrence properties), then the chain is positive recurrent if and only if the expected return time to the atom is finite.*

8 Challenges with Null Recurrence

Null recurrent Markov chains represent an interesting middle ground: they guarantee returns to each state (with probability 1), but the expected return times are infinite. Such chains lack stationary distributions, as shown by the following result:

Proposition 7. *For an irreducible Markov chain, the following are equivalent:*

1. *The chain is positive recurrent.*
2. *There exists a stationary distribution.*
3. *For some (and hence all) states i, $\sum_{n=1}^{\infty} P^n(i,i)$ diverges and $\liminf_{n\to\infty} nP^n(i,i) > 0$.*

This result highlights that null recurrent chains (where $P^n(i,i) \sim 1/\sqrt{n}$ for simple random walks) cannot support stationary distributions.

9 Conclusion

The Foster-Lyapunov criterion provides a powerful and general approach to establish positive recurrence and, consequently, the existence of stationary distributions for Markov chains with infinite state spaces. By carefully selecting a suitable Lyapunov function, one can analyze complex processes such as queueing systems, random walks, and general birth-death processes.

Key insights from this analysis include:

- Positive recurrence is necessary and sufficient for the existence of a stationary distribution in irreducible chains.

- The Foster-Lyapunov criterion provides a practical method to verify positive recurrence through drift conditions.
- Null recurrent chains, while ensuring returns to each state, do not possess stationary distributions due to infinite expected return times.
- For birth-death processes, explicit criteria and formulas for stationary distributions are available.

These results form the foundation for analyzing the long-term behavior of many practical systems with countably infinite state spaces.

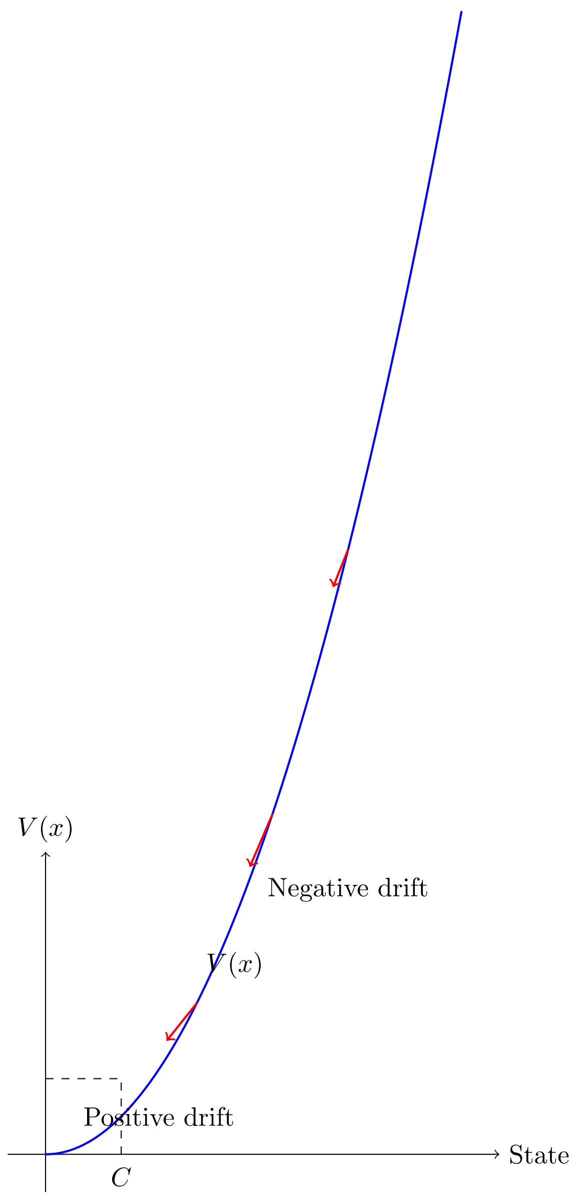

Figure 1: Illustration of the drift concept. Outside the set C, the Lyapunov function has negative drift (red arrows pointing downward), ensuring that the process does not escape to infinity.