# Existence of Stationary Distributions in Finite-State Markov Chains

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## 1 Introduction

A stationary distribution of a Markov chain represents a probability distribution that remains unchanged under the transition dynamics. In this document, we prove that a finite-state Markov chain that is irreducible and positive recurrent has a unique stationary distribution. We also discuss the role of aperiodicity in ensuring convergence to this stationary distribution.

## 2 Definition of Stationary Distribution

Let $\{X_t\}_{t\geq 0}$ be a time-homogeneous Markov chain with a finite state space $\mathcal{S} = \{1, 2, \ldots, N\}$ and transition probability matrix $P = (P_{ij})$. A probability distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ is called a **stationary distribution** if it satisfies

$$\pi P = \pi, \tag{1}$$

or equivalently,

$$\pi_j = \sum_{i=1}^{N} \pi_i P_{ij}, \quad \forall j \in \mathcal{S}, \tag{2}$$

and the normalization condition

$$\sum_{i=1}^{N} \pi_i = 1. \tag{3}$$

## 3 Key Definitions and Properties

**Definition 1** (Irreducibility). *A Markov chain is irreducible if for all $i, j \in \mathcal{S}$, there exists an integer $n > 0$ such that $P^n(i,j) > 0$. Equivalently, every state can be reached from any other state with positive probability in a finite number of steps.*

**Definition 2** (Positive Recurrence). *A state $i$ is positive recurrent if the expected return time to state $i$ is finite, i.e., $E[T_i] < \infty$, where $T_i = \min\{t \geq 1 : X_t = i \mid X_0 = i\}$.*

**Definition 3** (Aperiodicity)**.** *The period of a state $i$ is defined as $d(i) = \gcd\{n \geq 1 : P^n(i,i) > 0\}$, where* gcd *denotes the greatest common divisor. A state is aperiodic if $d(i) = 1$. A Markov chain is aperiodic if all its states are aperiodic.*

**Definition 4** (Ergodicity)**.** *A Markov chain is ergodic if it is irreducible, positive recurrent, and aperiodic.*

# 4 Existence of a Stationary Distribution

We establish the following theorem:

**Theorem 1.** *If a finite-state Markov chain is **irreducible** and **positive recurrent**, then it has a unique stationary distribution $\pi$ that satisfies*

$$\pi_j = \sum_{i=1}^{N} \pi_i P_{ij}, \quad \forall j \in \mathcal{S}. \tag{4}$$

*Moreover, $\pi_i > 0$ for all $i \in \mathcal{S}$.*

## 4.1 Proof Using the Perron-Frobenius Theorem

We first state the Perron-Frobenius theorem, which is central to our proof:

**Theorem 2** (Perron-Frobenius)**.** *Let $A$ be a non-negative, irreducible matrix. Then:*

1. *$A$ has a unique largest real eigenvalue $\lambda^* > 0$ (called the Perron root).*
2. *The corresponding left and right eigenvectors can be chosen to have strictly positive components.*
3. *Any other eigenvalue $\lambda$ satisfies $|\lambda| \leq \lambda^*$.*

Now we proceed with the proof of our main theorem:

1. The transition matrix $P$ of an irreducible Markov chain is a non-negative, irreducible matrix.
2. Since $P$ is a stochastic matrix (each row sums to 1), it has an eigenvalue $\lambda = 1$.
3. By the Perron-Frobenius theorem, this eigenvalue has a corresponding left eigenvector $\pi$ with strictly positive components satisfying $\pi P = \pi$.
4. After normalizing $\pi$ so that $\sum_{i=1}^{N} \pi_i = 1$, we obtain a probability vector that satisfies the definition of a stationary distribution.
5. Uniqueness follows from the Perron-Frobenius theorem, which guarantees that the eigenspace associated with the Perron root is one-dimensional.

### 4.2 Proof Using Long-Run Proportions

We now provide an alternative proof that gives more intuition about the meaning of the stationary distribution.

**Definition 5** (Indicator Function)**.** *For any event $E$, the indicator function $\mathbf{1}(E)$ equals 1 if $E$ occurs and 0 otherwise.*

For an irreducible, positive recurrent Markov chain, we can define $\pi_j$ as the long-run proportion of time spent in state $j$:

$$\pi_j = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}(X_t = j), \tag{5}$$

This limit exists almost surely and is independent of the initial state due to the irreducibility and positive recurrence of the chain.

To verify that $\pi$ satisfies the stationary equation, we consider the proportion of transitions into state $j$:

$$\begin{aligned}
\pi_j &= \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}(X_t = j) & (6)\\
&= \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathbf{1}(X_{t-1} = i) \cdot P_{ij} & (7)\\
&= \sum_{i=1}^{N} \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}(X_{t-1} = i) \cdot P_{ij} & (8)\\
&= \sum_{i=1}^{N} \pi_i P_{ij} & (9)
\end{aligned}$$

The second line follows from the Markov property, and the interchange of limit and summation in the third line is justified by the finite state space.

# 5 Convergence to the Stationary Distribution

While irreducibility and positive recurrence ensure the existence of a unique stationary distribution, aperiodicity is necessary for guaranteeing convergence of the distribution of $X_t$ to the stationary distribution as $t \to \infty$.

**Theorem 3** (Convergence Theorem)**.** *If a finite-state Markov chain is irreducible, positive recurrent, and aperiodic (i.e., ergodic), then for any initial distribution:*

$$\lim_{t\to\infty} P(X_t = j) = \pi_j, \quad \forall j \in \mathcal{S}. \tag{10}$$

This is a stronger result than the existence of long-run proportions, as it states that the probability distribution of the chain converges to the stationary distribution regardless of the initial state.

**Corollary 1.** *For an ergodic Markov chain with transition matrix $P$ and any initial distribution:*

$$\lim_{t \to \infty} P^t = \mathbf{1}\pi^T, \tag{11}$$

*where* $\mathbf{1}$ *is a column vector of ones and* $\pi^T$ *is the transpose of the stationary distribution vector.*

## 6 Extension to Countable State Spaces

The results presented above extend naturally to Markov chains with countably infinite state spaces. However, in the countable case:

1. Irreducibility and positive recurrence are still sufficient for the existence of a unique stationary distribution.

2. The proof using Perron-Frobenius needs to be replaced with more general arguments, as the theorem applies specifically to finite-dimensional matrices.

3. The long-run proportion approach remains valid, but requires more careful treatment of limit exchanges.

4. A key difference is that not all irreducible Markov chains with countable state spaces are positive recurrent; some may be null recurrent or transient.

## 7 Conclusion

We have shown that for a finite-state Markov chain that is irreducible and positively recurrent, a unique stationary distribution exists. Furthermore, we have clarified the role of aperiodicity in ensuring convergence to this stationary distribution. These results form the foundation of Markov chain theory and have widespread applications in economics, physics, and statistical mechanics.